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LETTER TO THE EDITOR

Statistical properties of low-connectivity bonds in percolation clusters

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Abstract. Numerical simulation of two-dimensional bond percolation at the percolation threshold shows that configurations with no singly connected bonds (SCB) appear with a finite probability. Nevertheless, the average minimal width of the percolation channels is small. This result confirms an assumption used to derive the critical exponent of the critical current density in superconductor-insulator composites. Higher-order bonds are shown to play the role of SCB in configurations with small numbers of SCB. Implications of the properties of low-connectivity bonds on the physical properties of percolating systems are discussed.

Finite-size scaling simplifies the treatment of the physical properties of percolating systems, since the geometrical properties of the percolation clusters on the length scales L smaller than the connectivity correlation length ξ can be conveniently classified and investigated. For instance, the critical current density j_c in a superconductor-insulator composite decays as $(p-p_c)^v$ when the volume fraction p of the superconducting component approaches the percolation threshold p_c. The numerical value of the exponent v has received considerable experimental and theoretical attention (see e.g. [1]). (A similar effect can be expected for the critical flux of superfluid He through a porous medium [2].) This singular behaviour of j_c is related to the divergence of the correlation length ξ near the threshold as $(p-p_c)^{-\nu}$. Simple finite-size scaling considerations show that for $L \ll \xi$, the critical current density $j_c \sim L^{-v/\nu}$. Thus, the problem is reduced to the investigation of j_c in the scaling, or fractal [3], regime $L \ll \xi$. It is convenient to reformulate the above statements in the language of the large-cell position space renormalisation group (LCRG) [4]: we will treat the probability $p'(p, L)$ that a given square (d-dimensional hypercubic) cell of linear size L percolates in one (predetermined) direction as the renormalised probability of a bond to be present after the original problem has been rescaled by a factor L. The fixed point $p_*(L)$ of such a transformation is determined from $p'(p_*, L) = p_*$, while the (effective) correlation length exponent ν is determined from the equation $L^{1/\nu} \equiv (\partial p'/\partial p)_{|p_*}$. The value of the (effective) exponent v can be determined from $j_c(p_*, L) \equiv j_c(p_*, 1)L^{-v/\nu}$. For $L \to \infty$, the exponents ν and v approach their asymptotic values, and $p_*(L) \to p_c$. All the simulations presented in this letter have been performed on a two-dimensional square lattice, which is self-dual and therefore $1-p'(p, L) = p'(1-p, L)$ and $p_*(L) = p_c = 0.5$. We must keep in mind that LCRG treatment of the *physical* properties of percolating systems presumes that (a) those properties have only one important length scale, which is proportional to the *geometrical* correlation length ξ, (b) the results are not sensitive

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to the choice of boundary conditions and (*c*) the results are not sensitive to the choice of the variable which is averaged and to the particular averaging procedure.

The simple 'nodes and links' model of Skal and Shklovskiĭ [5] and de Gennes [6], restated in the LCRG language, depicts the current-carrying backbone of the percolating cluster (at p_*, for finite L) as a single tortuous channel. It has been modified by Stanley [7], who observed that only a small portion of all bonds called 'singly connected bonds' (SCB) play a crucial role in the connectivity of the sample. A bond of a percolating configuration is called 'singly connected' if its removal creates a non-percolating configuration. Other low-connectivity bonds (LCB) can be defined in a similar manner. For example, a pair of bonds will be called 'doubly connected' if their simultaneous removal creates a non-percolating configuration, while none of them is a SCB. LCB play an important role in a variety of physical properties: they are responsible for the decay of spin-spin correlations of the Ising model in a percolating system at low temperatures [8]. The disappearance of the superconductivity in a superconductor-insulator composite is also caused by the high density currents flowing in the LCB, which cause the entire sample to become normal if the current density exceeds some threshold value. Knowledge of the average number of SCB can also be used to derive bounds on the critical exponent of elasticity [9] or to compare the critical behaviour of lattice models with the behaviour of continuum percolation models [10]. The possibility to infer physical properties from purely geometrical properties of the percolation clusters stimulated numerical investigations in that area (see e.g. [11]). However, the amount of exact statements which can be made about the geometry of the percolation clusters is quite limited. It can be shown [12, 13] that the average number of SCB in a hypercube of size L at p_c is proportional to $L^{1/\nu}$. If, following [12], we expand $p'(p_*-p, L)$ in powers of p and identify the coefficients of the expansion with the geometrical features of the percolating cluster, we find $p'(p_*-p, L)=p_*-\langle L_1\rangle p-\langle \frac{1}{2}L_1(L_1-1)-N_2\rangle (p)^2/p_*+\ldots$, where L_1 is the number of SCB, N_2 is the number of pairs of doubly connected bonds and $\langle\ldots\rangle$ denotes an ensemble average over all connected configurations. Thus $(\partial p'/\partial p)_{|p_*}=\langle L_1\rangle$, and therefore the relation $\langle L_1\rangle = L^{1/\nu}$ is a direct consequence of the definition of ν. Similar relations can be found for the higher-order derivatives of p'. These properties provide us with information concerning the averages, such as $\langle L_1\rangle$. However, in some cases the entire distributions may be important: treatment of spin-spin correlations of Ising spins requires ensemble averages of such quantities as $\exp(AL_1)$ (A is a temperature-dependent constant), which requires knowledge of various moments of L_1. Knowledge of the distributions of LCB may also clarify their role in such processes as conductivity and elasticity of percolating systems. In some cases various moments of LCB are interrelated. For example, the two-dimensional square lattice used in our simulations is self-dual, and therefore $(\partial^2 p'/\partial p^2)_{|p_*}=0$. Thus from the above expansion of $p'(p_*-p, L)$ we obtain the relation $2\langle N_2\rangle=\langle L_1^2\rangle-\langle L_1\rangle$. Consequently, the distribution function of L_1 also provides information on N_2, the direct evaluation of which is a very tedious numerical procedure.

A detailed knowledge of the statistics of LCB may also cast some light on the value of the exponent v describing the behaviour of j_c. Within the Skal-Shklovskiĭ-de Gennes model the entire superconducting current flows through a single channel and therefore, if the maximal current which can flow through a bond is I_0, we immediately find $j_c=I_0/L^{d-1}$, where L^{d-1} is the cross section of the hypercube. Comparing this expression with the definition of v, we find $v=(d-1)\nu$. Deutscher and Rappaport have noted [14] that a single-channel model is a poor representation of the percolation

backbone at low d, but the expression $v=(d-1)\nu$ may, nevertheless, remain valid if the system has (for $L \ll \xi$) at least one 'bottleneck'—a single bond through which the entire current must flow. Validity of the expression for v is consistent with both experimental results [14] and with numerical simulations of a simple model [15], but they cannot exclude the possibility of slow divergence of the width of the 'bottleneck' with increasing L. If at least one SCB is present in a sample it will be the 'bottleneck' mentioned above. However, the fact that $\langle L_1 \rangle$ diverges when $L \to \infty$ does not imply the presence of SCB in every sample, and more detailed information is required.

The following results have been obtained by computer simulation of the bond percolation at p_c on a two-dimensional $L \times L$ square lattice, for $L=5,\ldots,40, 80$. For each L, 40 000 configurations have been generated and the statistical properties of LCB is the connected configurations (half of all the configurations) have been analysed. The probability of having a sample with exactly L_1 SCB, $P_s(L_1)$, depends on the sample size L. We may expect it to have a scaling form $\langle L_1 \rangle^{-1}(L_1/\langle L_1 \rangle)$, where the function is independent of L. Indeed, superposition of plots of $P_s(L_1)\langle L_1 \rangle$ against $L_1/\langle L_1 \rangle$ for different L shows a reasonable convergence of the curves. Figure 1 depicts two such graphs (for $L=20$ and $L=80$). Only a slight difference can be seen. The convergence

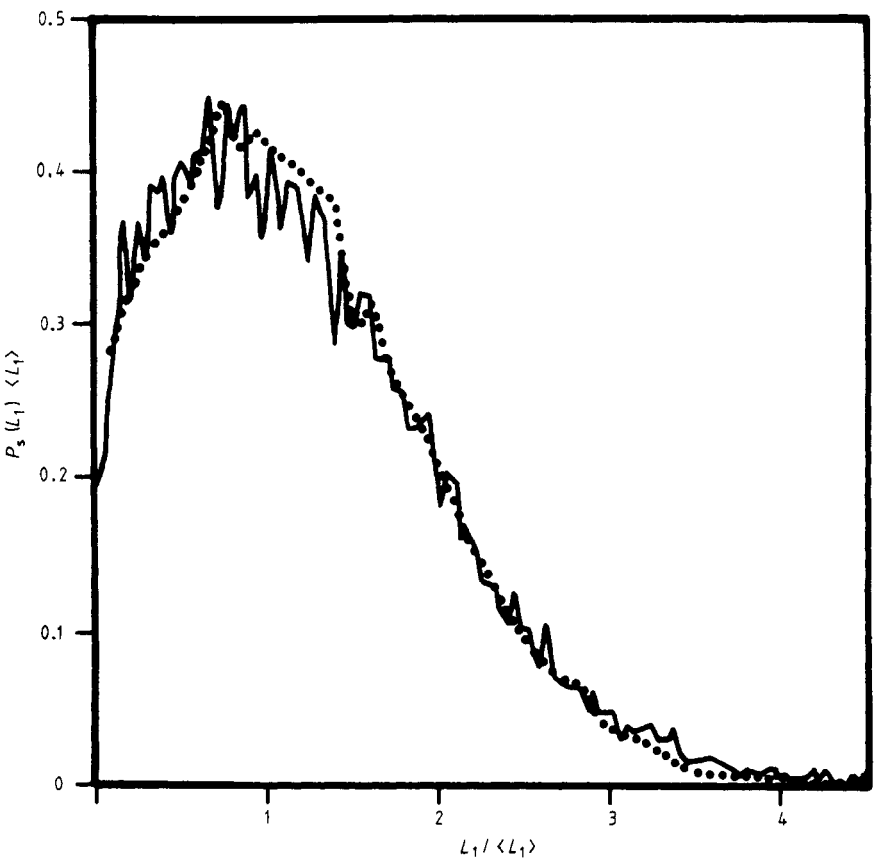

Figure 1. Probability distribution of the number of SCB (in scaled variables) for cell sizes $L=20$ (dotted curve) and $L=80$ (full curve). Only the probabilities for $L_1 \geqslant 1$ are shown.

is demonstrated more qualitatively in figure 2, which shows the dependence of the relative second and third moments of L_1 on L. The second moment seems to be well converged, while the third moment appears just to reach its finite value. The graphs in figure 1 depict only the results for $L_1 \geq 1$. While the probability to obtain any particular value of L_1 decays with increasing L, the configurations with no SCB have a constant probability $P_s(0) \approx 0.19$. This result casts certain doubts on the validity of the derivation of the expression for the exponent v, which assumed that the minimal width n of the channel is always one. In order to check the probabilities $P_w(n)$ that the minimal width is n, I performed the following procedure. If a percolating configuration had no SCB ($L_1 = 0$), I trimmed all the perimeter bonds on one side of the percolating cluster and checked the number L_2 of SCB in the resulting configuration.

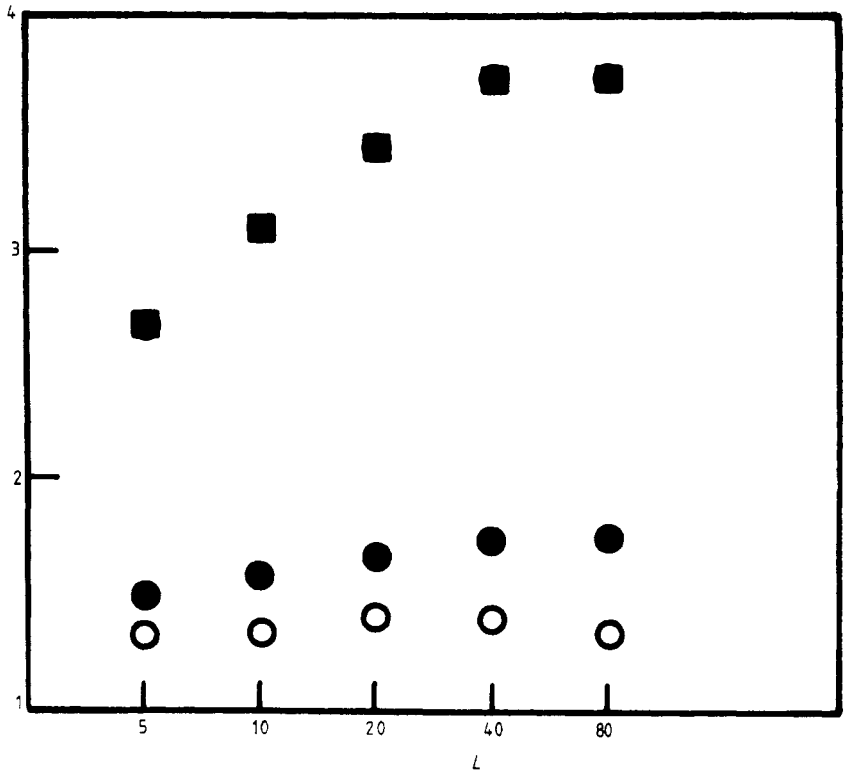

Figure 2. Semi-logarithmic plot of the relative second and third moments (middle and top, respectively) of the number of SCB L_1, and (half) of the number of doubly connected bonds L_2 (see text), when $L_1 = 0$, against the cell size L. ○, $\langle L_2 \rangle / \langle L_1 \rangle$; ●, $\langle L_1^2 \rangle / \langle L_1 \rangle^2$; ■, $\langle L_1^3 \rangle / \langle L_1 \rangle^3$.

(These SCB belonged to the pairs of doubly connected bonds in the original configuration.) If $L_2 > 0$, that meant that $n = 2$. In the case of $L_2 = 0$, the procedure has been repeated, and the number of SCB, L_3, in the new resulting configuration has been found, etc. The probabilities $P_w(n)$ for $n = 2$, 3, 4 are almost independent of L, as depicted in figure 3. The average width of the percolation channel was 1.20 ± 0.03, which is in excellent agreement with [16]. The probabilities $P_w(n)$ decay faster than

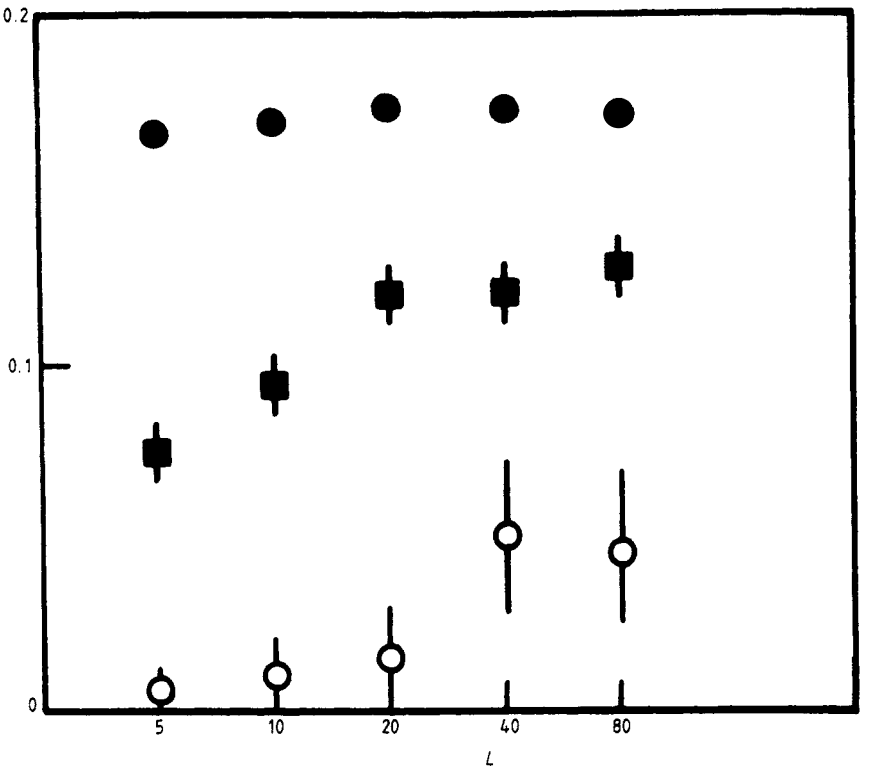

Figure 3. Semi-logarithmic plot of the probabilities $P_w(n)$ that the minimal width of the channel is n against the cell size L. Note that the values of the data points for $n=3$ and $n=4$ have been scaled. ●, $P_w(2)$; ■, $10P_w(3)$; ○, $100P_w(4)$.

exponentially with increasing n (approximately as $\exp(-n^{1.3})$): among 20 000 configurations only a few had $n=4$, and none had $n \geq 5$. This result suffices to restore the validity of the arguments, which lead to the expression for v. Although n is not necessarily one, the variability of the minimal width is extremely small and the results which have been obtained assuming $n=1$ remain valid. This result also means that the differences resulting from the application of different boundary conditions are insignificant. For example, in the problem of critical current we can either apply a fixed phase difference at the boundaries of the sample or assume that a certain current is forced through the sample. This might lead to different results if n had a wide distribution, but now it is clear that both cases will give essentially the same result.

The small variability of n also implies that the value of j_c is insensitive to the particular choice of equations: in the numerical simulation [15], j_c has been found by simply assuming that the currents in the sample will have an optimal distribution, which carries the maximal possible current across the sample, subject to the current conservation laws and the restriction that the current cannot exceed a certain threshold value in any of the conducting bonds. A more realistic superconductor–insulator network model could lead to a current distribution which is not necessarily optimal. However, the result would not differ by more than a prefactor from the results of [15], due to the presence of a 'bottleneck' of finite width. This conclusion should not be

extended to strongly non-linear models, such as the random fuse model [17], which has been suggested to simulate the rupture of brittle solids. This model resembles the random resistor network with one essential difference: if the current in a resistor exceeds a certain threshold value it 'burns', i.e. it is removed from the network, and the potentials are equilibrated again. This simulates crack propagation in a disordered solid, where the critical stress is represented by a critical value of the applied electric field. While the possibility of crack propagation creates a major difference between the maximal current density in the random fuse model and the superconductor-insulator network, on the length scales $L \ll \xi$ the problems coincide, since there is essentially only one bond to burn, and the crack propagation property cannot express itself. Such a simplistic argument could lead to the conclusion that the exponents v of the random fuse model and superconductor-insulator network coincide, but this is inconsistent with the numerical results obtained in [17]. The apparent contradiction may be resolved by assuming that crack propagation establishes an additional length scale which is not proportional to ξ of the percolation problem.

The probability distribution of L_2 (measured for $n=2$) is qualitatively similar to the distribution depicted in figure 1. (According to our definition, L_2 is proportional to the number of doubly connected bonds (on the average it is equal to half that number) in configurations with no SCB. It should not be confused with N_2, which measures the number of pairs of doubly connected bonds in all the connected configurations.) Moreover, as can be seen from figure 2, $\langle L_2 \rangle \sim \langle L_1 \rangle$. Similar statements can be made for L_3 and L_4. This provides some clues regarding the distributions of various physical quantities and their inverses: we know that the average resistance $\langle R \rangle$ and the average conductance $\langle\ \rangle$ of the connected configurations (at p_*) are related by $\langle R \rangle \sim \langle\ \rangle^{-1}$ [18]. This means that, although R has a wide distribution, the probabilities must decay sufficiently fast when $R \to 0$ (see [19]). SCB as well as other LCB play an important role in the resistance of the sample and we could expect the configurations with small L_1 to have low resistance. However, the probabilities of SCB do not decay for small L_1. Our results explain the apparent contradiction: since $\langle L_1 \rangle \sim \langle L_2 \rangle \sim \langle L_3 \rangle \sim \ldots$, we may conclude that when L_1 is small the role of SCB 'is taken over' by doubly connected bonds, etc.

In this letter I have described several statistical properties of LCB. The detailed analysis showed that, although the minimal width of percolation channels is not necessarily one, models which make this assumption lead to essentially correct results. The results also indicate that, despite extremely wide distributions of LCB, their collective effect on the physical properties may produce considerably smaller fluctuations. It shows that if we want to proceed beyond the simple properties which can be obtained using $\langle L_1 \rangle$, we need to have more detailed information about the correlations between the numbers of LCB of various orders.

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